

Multicontext logic for semigroups of contexts

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Abstract. A multicontext logic with algebraic structure is proposed, where contexts are either primitive or composed from other contexts. Composition of two contexts can support various intuitions: sequence concatenation, set union, multiset union, etc.

A local models semantics for algebraic context composition is defined, with a corresponding deductive calculus containing multilanguage bridge rules. Soundness and completeness results are proved for the case of semigroups of contexts, i.e. where context composition is an associative operation. Other properties of context composition, besides associativity, are defined by additional algebraic equations.

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1 Introduction

The Multi-Context Systems (MCS) formalism for contextual reasoning was proposed in [9] (under the name of Multi Language systems), and its semantics and proof theory have been developed in a series of papers [10, 8, 18]. MCSs are rich in vocabulary, having languages differentiated by context. They support the integration between reasoning *in* context and annotation *about* context. Most MCS defined in the literature presume some sort of structure of the context within which reasoning takes place, but here the richness varies a lot. In MCS there is a relatively simple indexing scheme which identifies context, and this has been extended to a partial order in some cases. Previous works do not provide a theoretical foundation for sets of contexts with a more complex structure.

However, in many applications, where contexts are used to represent pieces of knowledge that are dynamically modified, combined, copied, etc, it is useful to store (in the labels) the information about structural relations between context, as for instance, the fact that a context is a copy of another, or that a context is the union of other two different contexts. For this purpose we have to refine the labeling mechanisms so that we allow, for instance, the label $c \oplus d$ to denote a context obtained by combining the two contexts c and d .

In this paper, we are proposing MCSs with an enriched context structure, allowing contexts to combine into new composite contexts according to equationally specified patterns.

This paper is structured as follows: We start by giving two motivational examples, and then go on to introduce the algebraic notation and logical language we are using. Then, we define a semantics for interpreting formulas labeled by contexts, and a corresponding deductive calculus with inter-contextual deduction rules. Then a class of algebras called AFG algebras is introduced, and a completeness proof for that class is carried out. Finally, a comparison is made with some related work on formal models of contextual reasoning.

2 Partiality motivates algebraic contexts

Predicates about the world are partial, both in the absolute sense that no known theory covers everything in the universe, and also in the relative sense that usually, many facts which could have been expressed in the theory at hand are left unexpressed.

The term 'context' is used in a variety of senses in the literature, and we point out two competing ones:

In the philosophy of language, 'context' has been used to denote the total state of affairs which applies to an utterance being interpreted, while a projection onto selected facets of reality has been called an 'index' [13, 16].

This is contrast to emergent terminology in AI, where 'context' is almost invariably used to denote a partial state of affairs, and where the idea of a 'most general context', i.e. the context representing the total state of affairs, is sometimes denounced [14].

Without taking a metaphysical stand, we shall proceed to deal with representations of partial states of affairs, and refer to such representations as contexts. For the purposes of this paper, a context is either a primitive entity c (compare the 'micro-contexts' of [12, 11]), or a composite entity $x \oplus y$ obtained by combining other contexts x and y .

Whenever chunks of context combine into a larger joint context, issues of partiality, granularity, and specificity arise:

- what is the language of the joint context?
- what is true in the joint context?
- is one way of forming a joint context equivalent to another way?
- how do we move from one composite context to another?

In [2] it is suggested that contextual reasoning patterns come in three broad categories: localised reasoning, shifting, and push/pop. Localised reasoning is confined to one specific context, and shifting exchanges one context for another according to intercontextual rules. The third category, push/pop, includes patterns of reasoning where progression is from formulas asserted in one context to formulas asserted in an incrementally augmented/depleted context.

We propose to take an algebraic look at the operation of incrementally adding information to a context, or combining an existing context with another. Let us illustrate:

Example 1 *In this example, a context is defined by weather reports gathered from meteorological offices at airports. One important parameter for air traffic is horizontal visibility on the ground.*

In order for a scheduled flight to depart on time, it is necessary that certain weather minima are fulfilled, such as sufficient horizontal visibility for take-off and landing at the departure and destination airports.

A pilot planning a flight from Moscow via Paris to Washington must take into account three weather reports. Let us denote the contexts of local weather reports from each town by Mo, Pa, Wa , respectively. Beginning in context Mo , the pilot learns about Pa , and forms an accumulated context that we denote by

$$Mo \oplus Pa$$

Looking up the weather in Washington, this context is augmented further to

$$(Mo \oplus Pa) \oplus Wa$$

In this example, the order in which weather reports are looked up does not matter:

$$(Mo \oplus Pa) \oplus Wa = (Mo \oplus Wa) \oplus Pa$$

and retrieving the same weather report twice is equal to having it once:

$$Mo \oplus Mo = Mo$$

In fact, it seems adequate to think about a series of such amendments in terms of the set of increments that are made overall, so the \oplus operation in this example is a kind of set union. Set union is associative, commutative and idempotent, and the algebra of sets has equations corresponding to these three properties:

$$(u \oplus v) \oplus w = u \oplus (v \oplus w) \tag{1}$$

$$u \oplus v = v \oplus u \tag{2}$$

$$u \oplus u = u \tag{3}$$

In other examples (see below), there will be other algebraic equations defining other properties of context augmentation. Far from all context systems combine in the same way as sets do.

Before we leave this example, let us observe another trait having to do with translation between contexts: Visibility reports are routinely gathered and exchanged between airports worldwide. We can represent them as triples $\langle v, p, t \rangle$, where v is a measure of visibility, p is a geographical location, and t is a time-point. It is natural to think of these triples as members of a relation in a relational database, with attributes for visibility, location and time. When different

contexts are combined, triples from each component relation are put together in a joint relation. But, in a Russian database the visibility attribute would be called 'vidinost', and would be measured in meters, while in an American database the corresponding 'visibility' attribute would be measured in miles. The semantics of the \oplus operation should take this into account by providing a correspondence in the form of translations between entities of different local context languages.

This motivates a mapping which translates between local context languages, and later on this will be formalized. For more details of the formalization of federated databases in a multilanguage environment, see [8, 17].

Example 2 Let us take an example with a different flavour; the context defined by a person's beliefs. Here, the intuitive interpretation of the \oplus operation is the following. Given two contexts u and v , the context $u \oplus v$ is the context u extended with a description of v , (from u 's perspective). For instance if u is the context of John's beliefs, and suppose that John does not know about the existence of Mary. Let v be the context of Mary's beliefs. Suppose that John meets Mary, and then he has to do two things:

1. define a new context (denoted $u \oplus v$) representing John's view of Mary's beliefs;
2. relate this new context with his beliefs, i.e. with u .

This might involve things not expressed (or even expressible) in the original context u , for instance:

- extending the language. u does not contain any predicate for Mary's beliefs (this is the case if John does not know Mary), John has to add a new predicate (or modality) to express Mary's beliefs.
- extending the set of true formulae. John has to insert some new axioms expressing what he actually believes about Mary's beliefs.

With this intuitive interpretation we can stipulate epistemic variants of belief by admitting equations on \oplus terms. Admitting or rejecting equations like the following amounts to placement within an epistemic taxonomy.

For instance, we have that the idempotence equation

$$u \oplus u = u$$

states perfect introspection, while the associativity equation

$$(u \oplus v) \oplus w = u \oplus (v \oplus w)$$

means that for all u, v, w , from u 's point of view, v is a credible witness of w 's beliefs, i.e. mutual perfect introspection.

3 Algebras of contexts

Our concern in this paper is the context combination operator \oplus and its interaction with other features of MC formalism. We shall give a semantical structure

which is complete for certain intercontext deduction rules, when the properties of context combination are given as algebraic equations of a certain class. As shown by the previous examples, it is natural to think about algebraic concepts when two or more entities combine to form a new entity. If a countable set C of atomic contexts is given a priori, we can take it as the carrier of an algebra with one binary operator \oplus and equations

$$y_i = z_i, \quad 1 \leq i \leq N \quad (4)$$

for some $N > 0$, where y_i and z_i are terms on \oplus .

As an example consider bags (multisets) over C , generated by the \oplus operation. In a bag, as opposed to a sequence, the order of elements is immaterial. The relevant equations express associativity and commutativity of \oplus :

$$(u \oplus v) \oplus w = u \oplus (v \oplus w) \quad (5)$$

$$u \oplus v = v \oplus u \quad (6)$$

In a proper set over C , where repeated context entries don't count, the \oplus operation is also idempotent:

$$u \oplus u = u \quad (7)$$

We'll use E to denote the set of algebraic equations governing the \oplus symbol. The set of all \oplus -terms over C can be denoted C^\oplus , and the set of equivalence classes imposed by the algebraic equations E is then denoted C_E^\oplus .

Intuitively, C is the set of primitive labels for contexts. Each label $c \in C$ is associated with a context. Notice that different labels can be associated to the same context. The set C^\oplus is the set of *terms* for denoting contexts. Each $x \in C^\oplus$ is associated with a context, and, as for primitive labels, different terms can be associated to the same context. Finally, C_E^\oplus can be viewed as the set of "canonical names" for contexts. Each element $x \in C_E^\oplus$ is also associated with a context ctx , and x can be thought as a canonical name for ctx , or equivalently, the equivalence class (under the set of equation E) of the terms in C^\oplus associated with ctx . Notice that, for any $x, y \in C_E^\oplus$, if x is different from y then the context associated to x is different from the context associated to y . Keeping this in mind, in the rest of the paper, when it is not ambiguous, we use the term "context" as a shortcut of the description "label for context".

We proceed to define a semantics and a proof system for a large class of algebras of contexts. For simplicity of presentation we restrict ourselves to the case in which the languages associated to each context is propositional, and leave the first order case out for now.

3.1 Context languages

As a typographical convention, we use a, b, c, d , sometimes subscripted, to denote primitive contexts from C , while t, u, v, w, x, y, z and their subscripted variants

are used liberally to denote primitive contexts from C , composite contexts from C^\oplus , or equivalence classes of contexts from C_E^\oplus .

For each context $u \in C_E^\oplus$ we have a propositional language L_u , that is used to express facts in this context.

Definition 1 (Well formed formulae) *Well formed formulae are defined as follows (for all $u \in C_E^\oplus$). If ϕ is a propositional formula in L_u , then for all $y \in C^\oplus$ $y : \phi$ is a well formed formula, and ϕ is called a y -formula.*

Definition 2 (Language mapping) *For all $c \in C$, there is a partial recursive function l_c that maps $u \oplus c$ -formulae into u -formulae for arbitrary $u \in C_E^\oplus$.*

Intuitively, a language mapping from $u \oplus c$ to u states which part and how the content of the context $u \oplus c$ is represented in the context u . Considering the belief example (Example 2), the usual language mapping is the reification function, i.e. the total function that associates to each $u \oplus c$ -formula ϕ the u -formula $bel(c, \phi)$.

3.2 Local model semantics

Every equivalence class of contexts, i.e. each $u \in C_E^\oplus$, has its own formula language L_u . The semantical structure we are about to define, takes as its basic building blocks the local interpretations of each language L_u . We can identify interpretations with subsets of L_u , i.e. the true formulas in each interpretation.

The semantical structure for the entire system of languages reflects the way in which contexts are augmented by adding ground contexts by the \oplus operation. We start by defining ground extensions of context terms:

Definition 3 (x -continuation) *Given $x \in C_E^\oplus$, an x -continuation is a context*

$$(\dots(x \oplus c_1) \oplus c_2 \dots) \oplus c_h$$

where $0 \leq h$ and $c_i \in C$ for $1 \leq i \leq h$. When $h = 0$, this is just x . Note that unless \oplus is associative, the parentheses are not redundant.

Definition 4 (x -chain) *For $x \in C_E^\oplus$, an x -chain m is a function which maps every x -continuation y to a set m_y of interpretations of L_y (the local models of context y), such that for some x -continuation y , m_y is not empty, and for all x -continuations y , and ground contexts $c \in C$:*

- 1 $m \models y : l_c(\phi)$ if and only if $m \models y \oplus c : \phi$
- 2 the cardinality of m_y is at most 1.

Definition 5 (Satisfiability) *An x -chain m satisfies a formula $y : \phi$ where y is an x -continuation, in symbols $m \models y : \phi$, if for any $s \in m_y$, $s \models \phi$ according to the definition of satisfiability for propositional formulae.*

Definition 6 (Logical consequence) *A formula $x : \phi$ is a logical consequence of a set of formulae Γ , in symbols $\Gamma \models_E x : \phi$ if, for any z -chain m , such that x is an z -continuation, if $m \models \{y : \gamma \in \Gamma \mid x \neq_E y \text{ and } y \text{ is a } z\text{-continuation}\}$, then for all $s \in m_x$, $s \models \{\psi \mid z : \psi \in \Gamma, x =_E z\}$, implies that $s \models \phi$.*

3.3 Reasoning between contexts

The notion of x -continuations induces a partial order among contexts, each context preceding its continuations. Let us see how one moves between composite contexts which are related in a partial order. We rely on a natural deduction calculus extended with indices as described in [18], extended with the following bridge rules for any set of algebraic equations E .

$$\frac{u : l_c(\phi)}{u \oplus c : \phi} \text{Rup}_c \qquad \frac{u \oplus c : \phi}{u : l_c(\phi)} \text{Rdw}_c$$

$$\frac{u \oplus c : \phi \leftrightarrow \psi}{u : l_c(\phi) \leftrightarrow l_c(\psi)} \text{RRI} \qquad \frac{v : \phi}{u : \phi} u =_E v$$

Derivability from a set Γ of formulae can be defined from a set E of equations $\Gamma \vdash_E u : \phi$, if there is a deduction of $u : \phi$ from Γ , that uses u -rules (as defined in [18]) and the above bridge rules.

4 Semigroups with ground equations

Whenever associativity of the \oplus operation is given or implied, the resulting algebra is called a semigroup. In the present paper, we are restricting ourselves to certain semigroups of contexts called AFG algebras.

As shown in [15], AFG algebras have sets and bags (multisets) of contexts as special cases, as well as the 'flat contexts' of [4]. Technically, AFG algebras simplify the proof of completeness, and in the proof it is pointed out where associativity is used.

Definition 7 (AFG algebras) *An associative finite ground algebra, abbreviated AFG algebra, is one that satisfies these criteria:*

- *it is associative, i.e. contains the equation*

$$(u \oplus v) \oplus w = u \oplus (v \oplus w)$$

- *every equation apart from associativity is restricted so that the variables in it can only be instantiated by constants, not by terms containing \oplus .*
- *the number of equations is finite*

In semigroups, parentheses are redundant, so we may write for instance $(a \oplus b) \oplus (c \oplus d)$ as $a \oplus b \oplus c \oplus d$, or for that matter as $abcd$. The latter form is common in pure semigroups, i.e. where terms represent strings of atomic symbols and \oplus represents concatenation of strings.

There are many other AFG algebras besides strings, however, and we prefer to retain the generic \oplus symbol when writing terms from C^\oplus .

Observe that in sets, the axiom of commutativity can be restricted to context constants:

$$c \oplus d = d \oplus c \quad (8)$$

because with associativity we can get any permutation of

$$c_1 \dots c_m$$

by a series of applications of (8) on adjacent elements.

Idempotence can then be adequately taken as an AFG equation too:

$$c \oplus c = c \quad (9)$$

because with associativity and commutativity it is possible to collect equal elements of

$$c_1 \dots c_m$$

so that they are adjacent, and then repeatedly deleting an element where adjacent ones are equal by applying (9).

It is sometimes convenient to include a special context ϵ , such that

$$\epsilon \oplus u = u = u \oplus \epsilon \quad (10)$$

For example, in applications where there is an outermost supercontext, enclosing all other contexts, that could be ϵ .

5 Soundness and completeness

We can prove the following soundness and completeness result for AFG algebras:

Theorem 1 (Soundness and Completeness) *For any set of AFG equations E , $\Gamma \models_E u : \phi$ if and only if $\Gamma \vdash_E u : \phi$.*

5.1 Soundness

Soundness of Rdw_c and Rup_c are direct from item 1 of the chain conditions, and RRI is sound by virtue of item 2 of the chain conditions. Soundness of the remaining bridge rule

$$\frac{v : \phi}{u : \phi} u =_E v$$

follows because when $u =_E v$, any u -chain is also a v -chain.

5.2 Completeness

The completeness proof relies on canonical models which respect the bridge rules of our Multi Context system. The basic building blocks will be maximal consistent sets of well-formed formulae, adapted to the multilanguage environment. Let us state the versions of consistency and maximality that we need.

Definition 8 (*x*-consistency) *A finite set Δ of well-formed formulae is said to be *x*-consistent iff $\Delta \not\vdash_E x : \perp$, and an infinite set is *x*-consistent iff every finite subset is *x*-consistent.*

Definition 9 (*x*-maximality) *A set Δ of well-formed formulae is said to be *x*-maximal iff Δ is *x*-consistent and for all well-formed labelled formulae $y : \delta$ such that $\Delta \cup \{y : \delta\}$ is *x*-consistent, $y : \delta \in \Delta$.*

Theorem 2 (Lindenbaum) *Any *x*-consistent set of wffs can be extended to an *x*-maximal set.*

Proof: Start with an *x*-consistent set Δ_0 and an enumeration of all well-formed formulae $\langle x_i : \delta_i \rangle, i \geq 1$, and define inductively

$$\begin{aligned} \Delta_i &= \Delta_{i-1} \cup \{x_i : \delta_i\} \text{ if } \Delta_{i-1} \cup \{x_i : \delta_i\} \text{ is } x\text{-consistent} \\ &\Delta_{i-1} \text{ otherwise} \end{aligned} \quad (11)$$

Now

$$\Delta = \bigcup_{i=0}^{\infty} \Delta_i$$

is *x*-consistent, because otherwise by definition there would be a finite subset Δ^f such that $\Delta^f \vdash_E x : \perp$, and an index n such that $\Delta^f \subseteq \Delta_n$, contradicting *x*-consistency of Δ_n . To prove *x*-maximality, suppose $\Delta \cup \{x : \delta\}$ is *x*-consistent for some δ . Then $\Delta_{i-1} \cup \{x : \delta\}$ is also *x*-consistent, where i is the index of $x : \delta$ in the enumeration of wffs, therefore $x : \delta \in \Delta_i \subseteq \Delta$. \square

Canonical model Now let us choose an arbitrary *x*-consistent wff $x : \delta$ and construct an *x*-chain for it. To begin with, we expand $\{x : \delta\}$ to an *x*-maximal set Δ by the construction in the previous lemma.

Definition 10 (Canonical model) *For all *x*-continuations $y =_E x \oplus c_1 \dots \oplus c_h$*

- let $\Delta_y = \{\lambda \mid x : l_{c_1}(\dots l_{c_h}(\lambda)) \in \Delta\}$
- let S_y be the set of interpretations of the language L_y
- let $T_y = \{s \in S_y \mid s \models \Delta_y\}$ be the subset of interpretations that validate Δ_y
- and let m be the function that maps y to \emptyset if $T_y = \emptyset$ and to $\{t\}$ otherwise, where t is some arbitrary member of T_y .

*Our canonical model is the *x*-chain m .*

Δ_y is well-defined, so m is really an x -chain. To see this, we prove that

$$x : l_{c_1}(\dots l_{c_h}(\lambda)) \in \Delta \quad \text{iff} \quad x : l_{d_1}(\dots l_{d_k}(\lambda)) \in \Delta$$

whenever

$$x \oplus c_1 \dots \oplus c_h =_E x \oplus d_1 \dots \oplus d_k. \quad (12)$$

In fact,

$$x : l_{c_1}(\dots l_{c_h}(\lambda)) \in \Delta$$

iff, by h applications of *Rup*,

$$((x \oplus c_1) \dots \oplus c_h) : \lambda \in \Delta$$

iff, by associativity,³

$$x \oplus c_1 \oplus \dots \oplus c_h : \lambda \in \Delta$$

iff, by (12),

$$x \oplus d_1 \oplus \dots \oplus d_k : \lambda \in \Delta$$

iff, by associativity,

$$((x \oplus d_1) \dots \oplus d_k) : \lambda \in \Delta$$

iff, by k applications of *Rdw*,

$$x : l_{d_1}(\dots l_{d_k}(\lambda)) \in \Delta.$$

As regards the model conditions, the m we have defined here trivially fulfills condition 2, and condition 1 is fulfilled because for $c \in C$ and an x -continuation $y = x \oplus c_1 \oplus \dots \oplus c_h$, we have

$$y : l_c(\lambda) \in \Delta \quad \text{iff} \quad y \oplus c : \lambda \in \Delta$$

by *Rup*, *Rdw*, and x -maximality of Δ .

The x -chain m satisfies the wff $x : \delta$ (take $h = 0$), so we have completeness.

6 Related work

Algebras of contexts were proposed in [15] for single-language axiomatic context systems. In that work the modal logic with the *ist* modality defined in [12, 3, 7] is augmented with an algebra on the domain of contexts (the first argument of the *ist* modality). The results extend to arbitrary equational varieties. The *ist* formalism lacks much of the expressivity of a multilanguage formalism for context reasoning. The present work contains the first formalization of multicontext logics with algebraically structured contexts.

³ The construction does not depend essentially on associativity, so the results of this paper will be generalizable to arbitrary equational varieties by adjusting the definition of language mappings and using the initiality of term algebras. This is a work in progress.

A first proposal for an algebra of contexts is described in [6]. This work is more focused on the specific nature of the operation (the union) rather than a generic operation characterized via a set of equations, as it is here. On the other hand it seems that properties such as associativity, commutativity, idempotence, are relevant when one has to define operations between contexts.

A further theoretical framework, similar to a theory of contexts combination is proposed in [5], where contexts are formalized via a modal operator. The fact that ϕ is true in the context c is represented by the modal formula $\Box_c \phi$. Under this intuitive interpretation of modal formulas, combining contexts becomes combining modalities. An example of formula with a combined modality is $\Box_{c_1 \cup c_2} \phi$. This formula intuitively means that ϕ holds in the context $c_1 \cup c_2$, i.e., in the context obtained by combining c_1 and c_2 via a union operator. The main differences between this work and our approach are two. First, we have a general calculus for any operation definable via a set of equations, while [5] considers specific operations such as union, inverse, etc. Second, we deal with contexts with different languages, while in [5] all contexts share a common global language.

Algebras for contexts are relevant for the semantic web. The main assumption of the semantic web is that it is populated by a set of well-designed modular ontologies, each of which partially describes, from a specific perspective, a piece of knowledge. Constructing a new ontology is often a matter of assembling existing ones. Instead of building ontologies from scratch, one wants to reuse existing ontologies. Operations for combining ontologies are: ontology inclusion, ontology restriction, and polymorphic refinement. In [1], an argument is made in favour of encapsulating ontology in autonomous but partially coordinated contexts. Clearly to combine ontologies we need a way to combine contexts, which is what the present work is about.

7 Conclusion

In this paper we have described a general algebra for context combination in which the relation between source contexts and combined contexts is stored in the structure of the labels. This information was exploited semantically and proof-theoretically in order to infer relations between the content of different contexts. We have provided a semantics based on Local Model Semantics for the general case, and a proof method, based on Multi Context System for a limited, but significant, case, namely the set of contexts that are built via an operation axiomatisable by an AFG algebra.

The results of this paper can be lifted to other groupoids, and equational varieties in general, at the expense of slightly more complicated definitions and proofs. Furthermore, the results will generalize to local languages other than the propositional ones considered here, if only a minimal notion of satisfiability is defined. Work in these directions is in progress.

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